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THE EXISTENCE OF QUASIPERIODIC MOTIONS IN QUASILINEAR SYSTEMS†

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The qualitative methods of the Kolmogorov–Arnol'd–Moser theory are used to investigate a quasilinear oscillatory system with finite-dimensional frequency basis. The question of whether a perturbed system with the same basis has quasiperiodic solutions is formulated and studied, subject to suitable assumptions concerning the arithmetical properties of the characteristic indices of the generating system. The Bogolyubov–Mitropol'skii results are extended to the case in which the matrix of the linear system is non-singular and has purely imaginary eigenvalues. The existence of integral manifolds of a certain type is proved using the structural properties of the system, by almost identical transformations of the variables, the small parameter being scaled in a power sense. Besides the so-called algebraic critical case associated with the above-mentioned authors' work, some attention is devoted to the transcendental case of the critical part of the matrix, at the same time justifying and sharpening Moser's results in this area.

1. STATEMENT OF THE PROBLEM

Consider the n-dimensional quasilinear system

$$\dot{x} = \Lambda x + f(t) + \varepsilon X(t, x, \varepsilon) \tag{1.1}$$

whose right-hand side is a sufficiently smooth function for $0 \leqslant \varepsilon < \varepsilon^*$, $x \in D$, where D is a domain in $\mathbb{R}^n$. It is assumed that Λ is a constant matrix and that f and X are quasiperiodic as functions of t with a vector $\omega = (\omega_1, \ldots, \omega_m)$ of basis frequencies.

A function $z(t)$ is said to be quasiperiodic with basis frequencies $\omega_1, \ldots, \omega_m$ if these frequencies are rationally independent and a function $Z(\varphi)$, 2π-periodic in the components of the vector $\varphi = (\varphi_1, \ldots, \varphi_m)$ exists, such that $z(t) = Z(\omega_1 t, \ldots, \omega_m t)$. The function $Z(\varphi)$ will be called the generator of the quasiperiodic function $z(t)$. Henceforth we shall use the term "quasiperiodic functions" for functions with the same vector ω of basis frequencies. All functions of φ considered will be 2π-periodic in $\varphi_1, \ldots, \varphi_m$ and real analytic for $|\operatorname{Im}, \varphi| < \varphi^*$, $\varphi^* > 0$.

Our problem is whether quasiperiodic solutions of system (1.1) exist for small positive values of the parameter ε.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Λ. We know [1] that if $\operatorname{Re} \lambda_j \neq 0$ $(j = 1, \ldots, n)$, then system (1.1) has a unique quasiperiodic solution which tends, as $\varepsilon \to 0$, to a quasiperiodic solution of the generating equation

$$\dot{x} = \Lambda x + f(t) \tag{1.2}$$

We intend to extend this result, due to Bogolyubov, to the case of a matrix Λ which is non-singular but admits of pure imaginary eigenvalues.

2. ALGEBRAIC CASE

Let us assume that $\operatorname{Re} \lambda_j = 0$ $(j = 1, \ldots, 2l)$ and that none of the other eigenvalues of Λ lie on the imaginary axis. Assume, moreover, that the numbers $\lambda_j = i\alpha_j$ $(j = 1, \ldots, 2l)$ are distinct, non-zero and satisfy the Diophantine condition

$$\left|\sum_{i=1}^{m} q_i \omega_i + \sum_{i=1}^{2l} p_j \alpha_j\right| > \gamma\, |q|^{-\tau}, \quad \gamma > 0, \quad \tau \geqslant m \tag{2.1}$$

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where $|q| = |q_1| + \ldots + |q_m| > 0$, $|p| = |p_1| + \ldots + |p_{2l}| \leqslant 2$, $|p_1 + \ldots + p_{2l}| \leqslant 1$ (and p_j and q_j are integers). Condition (2.1) may be split into three systems of inequalities

$$\left|\sum_{i=1}^{m} q_i\omega_i\right| > \gamma|q|^{-\tau} \tag{2.2}$$

$$\left|\sum_{i=1}^{m} q_i\omega_i - \alpha_j\right| > \gamma|q|^{-\tau}, \quad j = 1,\ldots,2l \tag{2.3}$$

$$\left|\sum_{i=1}^{m} q_i\omega_i + \alpha_j - \alpha_k\right| > \gamma|q|^{-\tau}, \quad j,\ k = 1,\ldots,2l;\ j \neq k \tag{2.4}$$

Lemma 1. If conditions (2.3) hold, system (1.2) has a unique quasiperiodic solution $x = \psi(t)$.

Proof. Write system (1.2) in the form

$$\dot{\varphi} = \omega,\ \varphi(0) = 0;\ \dot{x} = \Lambda x + F(\varphi) \tag{2.5}$$

where $F(\varphi)$ is the generator of the quasiperiodic function $f(t)$. We must prove that system (2.5) has an integral manifold $x = \Psi(\varphi)$. That will be true if and only if the function $\Psi(\varphi)$ satisfies the equation

$$\frac{\partial \Psi}{\partial \varphi}\omega - \Lambda\Psi = F(\varphi)$$

But if condition (2.3) holds, this equation is uniquely solvable in the class of functions under consideration (see, for example, [2]).

Suppose that the trajectory $\psi(t)$ lies in the domain D. We shall seek a quasiperiodic solution of system (1.1) that tends to a generating solution $\psi(t)$ as $\varepsilon \to 0$. Set

$$y = x - \psi(t) \tag{2.6}$$

Then system (1.1) may be written as

$$\dot{y} = \Lambda y + \varepsilon X(t,\ y + \psi(t),\ \varepsilon) \tag{2.7}$$

If we put $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2l})$, system (2.7) may be written in suitable coordinates ξ, η in the form

$$\begin{aligned}
&\dot{\varphi} = \omega,\ \varphi(0) = 0\\
&\dot{\xi} = A\xi + \varepsilon a(\varphi) + \varepsilon T(\varphi)\xi + \varepsilon\Xi(\varphi,\ \xi,\ \eta,\ \varepsilon)\\
&\dot{\eta} = Q\eta + \varepsilon b(\varphi) + \varepsilon Y(\varphi,\ \xi,\ \eta,\ \varepsilon)\\
&\Xi(\varphi,\ 0,\ 0,\ 0) = 0,\quad Y(\varphi,\ 0,\ 0,\ 0) = 0,\quad \frac{\partial \Xi}{\partial \xi}(\varphi,\ 0,\ 0,\ 0) = 0
\end{aligned} \tag{2.8}$$

where Q is a non-critical matrix. The equations corresponding to complex conjugate elements of A will also be complex conjugates of one another.

Lemma 2. If condition (2.1) holds, a transformation of variables

$$\xi = u + \varepsilon p(\varphi) + \varepsilon P(\varphi)\, u,\quad \eta = \upsilon + \varepsilon q(\varphi) \tag{2.9}$$

exists which reduces system (2.8) to the form

$$\begin{aligned}
&\dot{\varphi} = \omega,\ \varphi(0) = 0\\
&\dot{u} = (A + \varepsilon B)u + \varepsilon U(\varphi,\ u,\ \upsilon,\ \varepsilon),\quad \dot{\upsilon} = Q\upsilon + \varepsilon V(\varphi,\ u,\ \upsilon,\ \varepsilon)\\
&U(\varphi,\ 0,\ 0,\ 0) = 0,\ V(\varphi,\ 0,\ 0,\ 0) = 0;\ \frac{\partial U}{\partial u}(\varphi,\ 0,\ 0,\ 0) = 0
\end{aligned} \tag{2.10}$$

where $B = \operatorname{diag}(B_1, \ldots, B_{2l})$ is a constant diagonal matrix, such that for complex conjugate pairs of

λ_j the corresponding B_j are also complex conjugates.

Proof. Differentiating equalities (2.9) along the trajectories of systems (2.8) and (2.10), we obtain equations for the vectors $p(\varphi)$, $q(\varphi)$ and the elements $P_{jk}(\varphi)$ of the matrix $P(\varphi)$

$$\begin{aligned} &\frac{\partial p}{\partial \varphi}\omega - Ap = a(\varphi),\ \frac{\partial q}{\partial \varphi}\omega - Qq = b(\varphi) \\ &\frac{\partial P_{jk}}{\partial \varphi}\omega + i(\alpha_j - \alpha_k)P_{jk} = T_{jk}(\varphi),\ j \neq k \\ &\frac{\partial P_{jj}}{\partial \varphi}\omega = T_{jj}(\varphi) - B_j \end{aligned} \tag{2.11}$$

If we define B_j to be the mean values of the functions $T_{jj}(\varphi)$ ($j = 1, \ldots, 2l$), then conditions (2.2)–(2.4) imply that Eqs (2.11) are solvable [2]. The assertion of the complex conjugate form of B_j follows from the complex conjugate nature of the functions $T_{jj}(\varphi)$.

Define

$$u = \sqrt{\varepsilon} z, \quad \upsilon = \sqrt{\varepsilon} w \tag{2.12}$$

System (2.10) takes the form

$$\begin{aligned} &\dot{\varphi} = \omega, \quad \varphi(0) = 0 \\ &\dot{z} = (A + \varepsilon B)z + \varepsilon^{3/2} Z(\varphi, z, w, \sqrt{\varepsilon}) \\ &\dot{w} = Qw + \sqrt{\varepsilon}\, W(\varphi, z, w, \sqrt{\varepsilon}) \end{aligned} \tag{2.13}$$

where Z and W are smooth functions for small $\|z\|$, $\|w\|$, $\sqrt{(\varepsilon)}$, and $\varphi \in \mathbb{R}^m$.

System (2.13) has an integral manifold of the form $z = z(\varphi, \varepsilon)$, $w = w(\varphi, \varepsilon)$[3, Lemma 2.1]. This implies the following theorem.

Theorem 1. If condition (2.1) holds and

$$\operatorname{Re} B_j \neq 0, \ j = 1, \ldots, 2l \tag{2.14}$$

then system (1.1) has a quasiperiodic solution for all sufficiently small $\varepsilon > 0$, which tends to a generating solution for $\varepsilon > 0$.

Theorem 1 is a special case of Theorem 2, to be proved below.

Remark. Since A has l pairs of purely imaginary eigenvalues, it follows that under suitable conditions system (1.1) also has—besides quasiperiodic solutions with m basis frequencies—invariant tori of arbitrary dimension from $m + 1$ to $m + l$ (see [2, Section 6, Chapter 1]). In that case bifurcation of the invariant torus will occur at $\varepsilon = 0$.

We will now consider a more general case.

Lemma 3. For any natural number ν, if condition (2.1) holds, a transformation

$$\xi = u + \varepsilon g(\varphi, u, \varepsilon), \ \eta = \upsilon + \varepsilon h(\varphi, u, \varepsilon) \tag{2.15}$$

exists which reduces system (2.8) to the form

$$\begin{aligned} &\dot{\varphi} = \omega, \quad \varphi(0) = 0 \\ &\dot{u} = (A + \varepsilon B^{(1)} + \ldots + \varepsilon^{\nu} B^{(\nu)})u + \varepsilon U(\varphi, u, \upsilon, \varepsilon) \\ &\dot{\upsilon} = Q\upsilon + \varepsilon V(\varphi, u, \upsilon, \varepsilon) \\ &U(\varphi, 0, 0, \varepsilon) = O(\varepsilon^{2\nu-1}), \quad V(\varphi, 0, 0, \varepsilon) = O(\varepsilon^{2\nu-1}) \\ &\frac{\partial U}{\partial u}(\varphi, 0, 0, \varepsilon) = O(\varepsilon^{\nu}), \quad \frac{\partial V}{\partial u}(\varphi, 0, 0, \varepsilon) = O(\varepsilon^{\nu-1}) \end{aligned} \tag{2.16}$$

where $B^{(j)} = \mathrm{diag}(B_1^{(j)}, \ldots, B_{2l}^{(j)})$ are constant diagonal matrices.

The proof is analogous to that of Lemma 2. The functions g and h in (2.15) are linear functions of u, whose coefficients and free terms are polynomials in ε.

We now put

$$u = \varepsilon^{\nu - 1/2} z, \quad \upsilon = \varepsilon^{2\nu - 1} w \tag{2.17}$$

Then system (2.16) takes the form

$$\begin{aligned} &\dot\varphi = \omega, \quad \varphi(0) = 0 \\ &\dot z = (A + \varepsilon B^{(1)} + \ldots + \varepsilon^{\nu} B^{(\nu)}) z + \varepsilon^{\nu + 1/2} Z(\varphi, z, w, \sqrt{\varepsilon}) \\ &\dot w = Qw + \sqrt{\varepsilon}\, W(\varphi, z, w, \sqrt{\varepsilon}) \end{aligned} \tag{2.18}$$

where Z and W are smooth functions for small $\|z\|$, $\|w\|$, $\sqrt{(\varepsilon)}$ and $\varphi \in \mathbb{R}^m$.

Theorem 2. If condition (2.1) holds and

$$(\mathrm{Re}\, B_j^{(1)})^2 + \ldots + (\mathrm{Re}\, B_j^{(\nu)})^2 > 0, \quad j = 1, \ldots, 2l \tag{2.19}$$

then system (1.1) has a quasiperiodic solution for sufficiently small $\varepsilon > 0$, which tends to a generating solution for $\varepsilon > 0$.

Proof. By condition (2.19), $A + \varepsilon B^{(1)} + \ldots + \varepsilon^{\nu} B^{(\nu)}$ is a diagonal matrix with blocks of the form $iS(\varepsilon) + \varepsilon^r R(\varepsilon)$, where $0 < r \leqslant \nu$, and $S(\varepsilon)$ and $R(\varepsilon)$ are real diagonal matrices such that $R(0)$ has non-zero diagonal elements. Suppose that there are just σ such blocks. System (2.18) can be reduced to the form

$$\begin{aligned} &\dot\varphi = \omega, \quad \varphi(0) = 0 \\ &\dot z_k = \varepsilon^{r_k}\left[iS_k(\varepsilon)\varepsilon^{-r_k} + R_k(\varepsilon)\right] + O(\varepsilon^{\nu + 1/2}) \\ &\dot w = Qw + O(\sqrt{\varepsilon}), \quad k = 1, \ldots, \sigma; \ 0 < r_k \leqslant \nu \end{aligned}$$

We may assume, without loss of generality, that $R_k = \mathrm{diag}\,(R_k^+, R_l^-)$, where the diagonal elements of the matrices R_k^+ are positive and the diagonal elements of R_k^- negative. Let $\varepsilon^{-r_k} S_k = \mathrm{diag}\,(S_k^{(1)}(\varepsilon), S_k^{(2)}(\varepsilon))$. Letting

$$J_k(t) = \begin{cases} -\mathrm{diag}(X_{1k}^+, 0), & t > 0 \\ \mathrm{diag}(0, X_{2k}^-), & t < 0 \end{cases}$$

$$\chi_{mk}^{\pm} = \exp\left[-(iS_k^{(m)} + R_k^{\pm})t\right], \quad m = 1, 2; \ k = 1, \ldots, \tau$$

we see that for all $t \in \mathbb{R}$

$$\|J_k(t)\| \leqslant \beta e^{-\alpha t}, \quad \alpha > 0, \quad \beta > 0$$

Using this estimate as in the proof of the theorem in [2, Section 3, Chapter 1], we can prove that system (2.18) has an integral manifold $z = z(\varphi, \varepsilon)$, $w = w(\varphi, \varepsilon)$. Substituting these expressions into (2.17), and using (2.14) and (2.6), as well as the fact that $\varphi = \omega t$, we obtain the assertion of the theorem.

If a natural number ν exists such that condition (2.19) holds, then, following Lyapunov, we say that the algebraic case is effective. Otherwise we speak of the transcendental case. We have thus been considering the algebraic case.

3. THE TRANSCENDENTAL CASE

Considering system (2.10), assume that

$$B_j = i\beta_j, \quad \beta_j \neq 0, \ j = 1, \ldots, 2l \tag{3.1}$$

where the numbers β_j are real and pairwise distinct.

We will confine our attention to the case when $n = 2l$, i.e. system (1.1) has no non-critical variables. Assume, moreover, that the functions on the right in (1.1) are real analytic for small $|\operatorname{Im} t|$ and $|\varepsilon|$, and analytic in x in a complex neighbourhood of the domain D. Accordingly, system (2.13) is analytic for small $\|z\|$, $\|w\|$, $\|\operatorname{Im}\varphi\|$, $|\sqrt{\varepsilon}|$.

The numbers $\lambda_j = i\alpha_j$, $B_j = i\beta_j$ split into complex conjugate pairs. Letting $\alpha_k = -\alpha_{k+l}$, $\beta_k = -\beta_{k+l}$ $(k = 1, \ldots, l)$, we may assume (without loss of generality) that $\beta_k > 0$.

Suppose $\mu_1, \ldots, \mu_{2l}$, where $\mu_k = -\mu_{k+l}$, are numbers that satisfy condition (2.1) with $\gamma = K\varepsilon$, i.e.

$$\left|\sum_{i=1}^{m} q_i\omega_i + \sum_{j=1}^{2l} p_j\mu_j\right| > K\varepsilon|q|^{-\tau}, \quad K>0, \quad \tau>0$$
$$|q|>0, \ |p|\leqslant 2, \ |p_1+\ldots+p_{2l}|\leqslant 1 \tag{3.2}$$

Define the matrix $M = \operatorname{diag}(i\mu_1, \ldots, i\mu_{2l})$.

The following assertion was established in [4].

Assertion. A number $\varepsilon^*(K) > 0$ exists with the following property: matrix-valued functions $\Delta(\sqrt{\varepsilon})$, $C(\varphi, \sqrt{\varepsilon})$ and a vector-valued function $c(\varphi, \sqrt{\varepsilon})$ exist, all analytic in $\sqrt{(\varepsilon)}$ for $|\sqrt{\varepsilon}| < \sqrt{\varepsilon^*}$, such that the transformation

$$z = \zeta + c(\varphi, \sqrt{\varepsilon}) + C(\varphi, \sqrt{\varepsilon})\zeta \tag{3.3}$$

reduces the system

$$\dot{\varphi} = \omega, \ \varphi(0) = 0; \ \dot{z} = (M + \varepsilon^{3/2}\Delta(\sqrt{\varepsilon}))z + \varepsilon^{3/2} Z(\varphi, z, \sqrt{\varepsilon}) \tag{3.4}$$

to the form

$$\dot{\varphi} = \omega, \ \varphi(0) = 0; \ \dot{\zeta} = M\zeta + Y(\varphi, \zeta, \sqrt{\varepsilon}) \tag{3.5}$$

$$\left(Y(\varphi, 0, \sqrt{\varepsilon}) = 0, \ \frac{\partial Y}{\partial \zeta}(\varphi, 0, \sqrt{\varepsilon}) = 0\right) \tag{3.6}$$

Moreover, $\Delta(\sqrt{\varepsilon})$ is a diagonal matrix, and complex conjugate elements of M correspond to complex conjugate elements of Δ.

Consider the question of the existence of numbers $\mu_1, \ldots, \mu_{2l}$ that satisfy condition (3.2). Take $\tau = m + 1$. Define $\mu_k - \alpha_k = \varepsilon_k$ $(k = 1, \ldots, l)$. If $|p| = 0$, inequality (3.2) is true by virtue of (2.2), since we may assume that $K\varepsilon < \gamma$. If $|p| = 1$, inequalities (3.2) may be written as

$$\left|\sum_{i=1}^{m} q_i\omega_i - (\alpha_k + \varepsilon_k)\right| > K\varepsilon|q|^{-(m+1)}, \quad k = 1, \ldots, l \tag{3.7}$$

If $|p| = 2$, inequalities (3.2) are

$$\left|\sum_{i=1}^{m} q_i\omega_i + (\alpha_j + \varepsilon_j) \mp (\alpha_k + \varepsilon_k)\right| > K\varepsilon|q|^{-(m+1)}$$
$$k, j = 1, \ldots, l; \ k \neq j \tag{3.8}$$

Lemma 4. For any $\theta > 0$, $\eta > 0$, one can choose $K(\theta, \eta) > 0$ so that the inequality

$$\left|\sum_{i=1}^{m} q_i\omega_i - \Omega\right| > K\varepsilon|q|^{-(m+1)} \tag{3.9}$$

will hold for values of Ω in some set Γ, such that for any interval L of length $2\eta\varepsilon$ it is true that mes $(\Gamma \cap L) > 2\eta\varepsilon(1 - \theta)$.

Proof.† Let us determine the measure of the set of all $\Omega > 0$ such that inequality (3.9) fails to hold, i.e.

$$\left|\sum_{i=1}^{m} q_i\omega_i - \Omega\right| \leq K\varepsilon|q|^{-(m+1)}$$

For given $q_1, \ldots, q_m$ the measure of the set of all such Ω is $2K\varepsilon\ |q|^{-(m+1)}$. This quantity must be summed over all integers $q_1, \ldots, q_m$. Since the number of different tuples $q_1, \ldots, q_m$ with the same norm $|q|$ is majorized by a quantity $c_1|q|^{m-1}$, $c_1 > 0$, it follows that the sum cannot exceed

$$\sum_{|q|=1}^{\infty} \frac{2c_1K_\varepsilon}{|q|^2} = c_2K_\varepsilon,\ \ c_2 > 0$$

Consequently, mes $(\Gamma \cap L) \geq 2\eta\varepsilon - c_2K\varepsilon$. Taking $K < 2\eta\theta c_2^{-1}$, we see that mes $(\Gamma \cap L) > 2\eta\varepsilon(1-\theta)$, which it was required to prove.

Corollary. For any $\eta_k > 0$ $(k = 1, \ldots, l)$, inequalities (3.7) will hold for $\varepsilon_k \in \Gamma_k \subset (-\eta_k\varepsilon, \eta_k\varepsilon)$, and moreover, by suitable choice of K, the measure of Γ_k may be made as close to $2\eta_k\varepsilon$ as desired. An analogous statement holds for numbers $(\varepsilon_j \mp \varepsilon_k \in \Gamma_{kj}^{\mp} \subset (-\eta_j + \eta_k)\varepsilon, (\eta_j + \eta_k)\varepsilon)$ satisfying inequalities (3.8).

Let us assume that

$$\Delta = i\,\mathrm{diag}(d_1,\ldots,d_l,\ -d_1,\ldots,\ -d_l) \tag{3.10}$$

where $d_k(\sqrt{\varepsilon})$ $(k = 1, \ldots, l)$ are real analytic functions.

Theorem 3. If conditions (2.1), (3.1) and (3.10) hold, then in any positive half-neighbourhood of $\varepsilon = 0$, a value of ε exists for which system (1.1) has quasiperiodic solutions that tend to a generating solution as $\varepsilon \to 0$. Moreover, for $\varepsilon \in (0, \varepsilon_0)$, the measure of the set of such values of ε is equivalent as an infinitesimal quantity to ε_0.

Proof. Consider system (3.5). It follows from (3.6) that this system has a solution $\varphi = \omega t$, $\zeta = 0$, to which, by (3.3), there corresponds a solution $\varphi = \omega t$, $z = c(\omega t, \sqrt{\varepsilon})$ of system (3.4). System (3.4) will be identical with system (2.13) (where, in this case, the third equation drops out) if

$$A + \varepsilon B = M + \varepsilon^{3/2}\Delta(\sqrt{\varepsilon}) \tag{3.11}$$

Fix $\varepsilon = \varepsilon_0$, where ε_0 is sufficiently small. Let $\varepsilon \in (0, \varepsilon_0)$, $\eta_k = \beta_k/2$. Fix K so that, as guaranteed by the corollary to Lemma 4, the measure of the sets Γ_k is sufficiently close to $2\eta_k\varepsilon_0$ for all $k = 1, \ldots, l$.

Since the matrix M depends on parameters $\varepsilon_1, \ldots, \varepsilon_l$, the function Δ depends not only on $\sqrt{\varepsilon}$ but also on these parameters. As a function of the parameters, it is defined by virtue of (3.7), (3.8) and the corollary to Lemma 4 on a certain set $\Gamma \subset \Pi_{k=1}^{l}(-\eta_k\varepsilon_0, \eta_k\varepsilon_0)$. In addition, it has been shown [5] that as a function of $\varepsilon_1, \ldots, \varepsilon_l$ it may be extended to a C^∞ function on the direct product of the intervals $(-\eta_k\varepsilon_0\, \eta_k\varepsilon_0)$. Denote the extended function by $E = i$ diag $(E_1, \ldots, E_l, -E_1, \ldots, -E_l)$.

We write Eq. (3.10) with the extended function Δ in terms of the coordinates

$$\beta_k\varepsilon - \varepsilon_k - \varepsilon^{3/2}E_k(\sqrt{\varepsilon},\ \varepsilon_1,\ldots,\varepsilon_l) = 0,\ \ k = 1,\ldots,l \tag{3.12}$$

The Jacobian of this system in the parameters $\varepsilon_1, \ldots, \varepsilon_l$ does not vanish at $\varepsilon = \varepsilon_1 = \ldots = \varepsilon_l = 0$. Hence system (3.12) has a solution $\varepsilon_k, = \beta_k\varepsilon + F_k(\varepsilon)$, $k = 1, \ldots, l$, with $F_k(\varepsilon) = o(\varepsilon)$. The functions F_k are defined for $\varepsilon \in (0, \varepsilon_0/2)$. However, we are interested only in ε values with

$$\begin{aligned} &\varepsilon_k, = \beta_k\,\varepsilon + F_k(\varepsilon) \in \Gamma_k \\ &\varepsilon_j \mp \varepsilon_k = (\beta_j \mp \beta_k) + (F_j(\varepsilon) \mp F_k(\varepsilon)) \in \Gamma_{kj}^{\mp} \\ &k, j = 1,\ \ldots,\ l \end{aligned} \tag{3.13}$$

†This proof is due to A. D. Bryuno.

By the corollary to Lemma 4, each of conditions (3.10) defines a set of values of $\varepsilon = (0, \varepsilon_0/2)$ whose measure is as close to $\varepsilon_0/2$ as desired. By additivity, the measure of the set of ε satisfying all conditions (3.13) is also as close to $\varepsilon_0/2$ as desired. For these ε values system (2.13) has a solution $\varphi = \omega t, z = c(\omega t, \sqrt{\varepsilon})$, and by (2.6), (2.9) and (2.12), this solution determines the desired quasiperiodic solution of system (1.1). This proves Theorem 3.

For a fixed matrix A, condition (2.1) holds for almost all frequency vectors ω. Conditions (3.1) and (3.10), on the contrary, impose substantial restrictions on the nature of the system. These restrictions are analogous to those imposed on a system of differential equations for KAM-theory to be applicable. It is well known that KAM-theory is applicable to Hamiltonian systems and reversible systems. We will show that conditions (3.1) and (3.10) are also satisfied for Hamiltonian and reversible systems. It will be shown simultaneously that Hamiltonian and reversible systems belong to the transcendental case.

We begin with reversible systems. System (1.1) is said to be reversible if the Fourier coefficients with respect to φ of the right-hand side of the corresponding system (2.8) with diagonal A are purely imaginary functions of the real variables ξ and ε (the variable η does not occur in (2.8)).

We will give an example of a reversible system. Consider the system of second-order differential equations

$$\ddot{x}_k + a_k^2 x_k = f_k(t) + \varepsilon X_k(t, x, \dot{x}, \varepsilon), \quad k = 1,\dots,d \tag{3.14}$$

where the functions f_k and X_k satisfy the conditions imposed in Section 3 on the coordinate functions of the vectors f and X in system (1.1).

Lemma 5. If

$$f_k(-t) = f_k(t), \; X_k(-t, x, -\dot{x}, \varepsilon) = X_k(t, x, \dot{x}, \varepsilon), \; k = 1,\dots,d \tag{3.15}$$

then system (3.14) is reversible.

Proof. The generating solution of system (3.14) is an even function. The system corresponding to (2.7) is

$$\dot{y}_{2k-1} = -ia_k y_{2k-1} + ia_k^{-1}\varepsilon g_k\left(t, \frac{1}{2}(y_{2j-1}+y_{2j}), \frac{a_j}{2i}(y_{2j-1}-y_{2j}), \varepsilon\right)$$

$$\dot{y}_{2k} = ia_k y_{2k} - ia_k^{-1}\varepsilon g_k\left(t, \frac{1}{2}(y_{2j-1}+y_{2j}), \frac{a_j}{2i}(y_{2j-1}-y_{2j}), \varepsilon\right)$$

$$g_k(t, u_j, \dot{u}_j, \varepsilon) = X_k(t, u_j + \sigma_j(t), \dot{u}_j + \dot{\sigma}_j(t), \varepsilon) - X_k(t, \sigma_j(t), \dot{\sigma}_j(t), \varepsilon)$$

$$j, k = 1, \dots, d$$

where $\sigma(t)$ is a generating solution. The functions

$$Y_k(\varphi, y, \dot{y}, \varepsilon) = a_k^{-1} G_k\left(\varphi, \frac{1}{2}(y_{2j-1}+y_{2j}), \frac{a_j}{2}(y_{2j-1}-y_{2j}), \varepsilon\right)$$

where G_k are generators of quasiperiodic functions g_k, which are invariant, by (3.15), under the transformation $i \to -i$, $\varphi \to -\varphi$. Hence their Fourier coefficients are self-conjugate, i.e. real.

Corollary. A quasilinear differential equation

$$x^{(2d)} + b_1 x^{(2d-2)} + \dots + b_d x = f(t) + \varepsilon X(t, x, \dot{x}, \dots x^{(2d-1)}, \varepsilon)$$

is reversible if all roots of the equation

$$\lambda^d + b_1\lambda^{d-1} + \dots + b_d = 0$$

are negative and moreover $f(-t) = f(t)$

$$X(-t, x, -\dot{x}, \dots, -x^{(2d-1)}, \varepsilon) = X(t, x, \dot{x}, \dots, x^{(2d-1)}, \varepsilon)$$

Theorem 4. If system (1.1) is reversible and condition (2.1) holds, the conclusion of Theorem 3 holds.

Proof. We must prove that conditions (3.1) and (3.10) hold for reversible systems. Condition (3.1) follows from the last equation of (2.11).

In addition, it follows from Eqs (2.11) that the Fourier coefficients of the coordinate functions $p(\varphi)$ and $P(\varphi)$ in the first equation of (2.9) are real. Expanding the right-hand side of (2.8) in power series in

$\xi = y$ and ε, and in Fourier series in φ, we obtain series with purely imaginary coefficients. Consequently, the same is true of system (2.10), hence also of (2.13). The transformation (3.3) is the composition of an infinite number of transformations of type (2.9) [4]. Thus condition (3.10) holds, since the elements of Δ are obtained by summing the free terms of Fourier series with the above-mentioned property.

We will now consider Hamiltonian systems.

Theorem 5. If system (1.1) is Hamiltonian and condition (2.1) holds, the conclusion of Theorem 3 holds.

Proof. The transformations of system (1.1) to system (2.7) are canonical. Consequently, adding the equation $\dot{r} = -\varepsilon\partial H/\partial r$ to system (2.8), where εH is a generator of the Hamiltonian of system (2.7) as a quasiperiodic function of t, we obtain a Hamiltonian system with Hamiltonian $\omega' r + \varepsilon H(\varphi, x, y, \varepsilon)$, where the pair (x, y) corresponds to ξ in (2.8) (the variable η does not occur).

We construct a transformation (2.9) as a canonical transformation of the variables φ, r, x and y into the variables $\varphi, \bar{r}, \bar{x}$ and $\bar{y}$ with generating function $\varphi' r + x' y + \varepsilon S(\varphi, \bar{x}, \bar{y}, \varepsilon)$. Since H does not depend on r, systems (2.10) and (2.13) are Hamiltonian. Hence the coefficients of u and z, respectively, on the right-hand sides differ only in sign from their complex conjugates, i.e. they are purely imaginary, so that (3.1) holds.

But (3.3) is a composition of such canonical transformations. Hence the elements of the matrix Λ are also purely imaginary, so that condition (3.10) also holds.

Remark. Following the proof that condition (3.1) holds, we can also prove that in Hamiltonian systems condition (2.19) does not hold for any j and ν.

Example. Consider the Duffing equation

$$\ddot{x} + a(\varepsilon)x = bx^3 + \varepsilon h(t), \quad a(\varepsilon) > 0, \quad \varepsilon > 0$$

Setting $x = \sqrt{(\varepsilon)}y$, we obtain a quasilinear equation

$$\ddot{y} + a(0)y = \sqrt{\varepsilon}\, h(t) + \varepsilon(by^3 + c(\varepsilon)y), \quad c(\varepsilon) = \varepsilon^{-1}(a(\varepsilon) - a(0)) \tag{3.16}$$

Since this equation may be represented in Hamiltonian form, Theorem 5 is applicable. Hence for small positive ε the typical situation for Eq. (3.16) is the existence of a quasiperiodic solution with the same frequency basis as $h(t)$.

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